

(2)

AD-A200 448

IDA PAPER P-2144

## ATTRITION PROCESSES WITH PARKING AREAS

Eleanor L. Schwartz

September 1988

DTIC  
ELECTED  
S OCT 06 1988  
H

DISTRIBUTION STATEMENT A

Approved for public release  
Distribution Unlimited



INSTITUTE FOR DEFENSE ANALYSES

1801 N. Beauregard Street, Alexandria, Virginia 22311-1772

88 10 5 082

IDA Log No. HQ 88-33717

## DEFINITIONS

IDA publishes the following documents to report the results of its work.

### Reports

Reports are the most authoritative and most carefully considered products IDA publishes. They normally embody results of major projects which (a) have a direct bearing on decisions affecting major programs, or (b) address issues of significant concern to the Executive Branch, the Congress and/or the public, or (c) address issues that have significant economic implications. IDA Reports are reviewed by outside panels of experts to ensure their high quality and relevance to the problems studied, and they are released by the President of IDA.

### Papers

Papers normally address relatively restricted technical or policy issues. They communicate the results of special analyses, interim reports or phases of a task, ad hoc or quick reaction work. Papers are reviewed to ensure that they meet standards similar to those expected of refereed papers in professional journals.

### Memorandum Reports

IDA Memorandum Reports are used for the convenience of the sponsors or the analysts to record substantive work done in quick reaction studies and major interactive technical support activities; to make available preliminary and tentative results of analyses or of working group and panel activities; to forward information that is essentially unanalyzed and unevaluated; or to make a record of conferences, meetings, or briefings, or of data developed in the course of an investigation. Review of Memorandum Reports is suited to their content and intended use.

The results of IDA work are also conveyed by briefings and informal memoranda to sponsors and others designated by the sponsors, when appropriate.

The work reported in this document was conducted under IDA's Independent Research Program. Its publication does not imply endorsement by the Department of Defense or any other government agency, nor should the contents be construed as reflecting the official position of any government agency.

This paper has been reviewed by IDA to assure that it meets high standards of thoroughness, objectivity, and sound analytical methodology and that the conclusions stem from the methodology.

Approved for public release; distribution unlimited.

## UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE				
1a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED		1b. RESTRICTIVE MARKINGS		
2a. SECURITY CLASSIFICATION AUTHORITY DD Form 1473, dated 1 October 1983.		3. DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A				
4. PERFORMING ORGANIZATION REPORT NUMBER (S) IDA Paper P-2144		5. MONITORING ORGANIZATION REPORT NUMBER (S)		
6a. NAME OF PERFORMING ORGANIZATION INSTITUTE FOR DEFENSE ANALYSES		6b. OFFICE SYMBOL (If applicable)	7a. NAME OF MONITORING ORGANIZATION OSD, OUSD(A) DoD-IDA Management Office	
6c. ADDRESS (City, State, and Zip Code) 1801 North Beauregard Street Alexandria, Virginia 22311		7b. ADDRESS (CITY, STATE, AND ZIP CODE) 1801 North Beauregard Street Alexandria, Virginia 22311		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION N/A		8b. OFFICE SYMBOL (If applicable)	9. PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER IDA Independent Research	
8c. ADDRESS (City, State, and Zip Code) N/A		10. SOURCE OF FUNDING NUMBERS		
		PROGRAM ELEMENT	PROJECT NO.	TASK NO. ACCESSION NO. WORK UNIT
11. TITLE (Include Security Classification) ATTRITION PROCESSES WITH PARKING AREAS				
12. PERSONAL AUTHOR (S). Eleanor L. Schwartz				
13. TYPE OF REPORT Final	13b. TIME COVERED FROM 12/87 TO 9/88	14. DATE OF REPORT (Year, Month, Day) September 1988		15. PAGE COUNT 38
16. SUPPLEMENTARY NOTATION				
17. COBALT CODES		18. SUBJECT TERMS (Continue on reverse if necessary and identify by block number) airbase attack, attrition equations, combat simulation, force effectiveness, mathematical modeling, probability, stochastic applications, warfare models		
19. ABSTRACT (Continue on reverse if necessary and identify by block number) This paper examines a class of combat processes in which targets can be located on "parking areas," so that an attack on a target can kill other targets on the same parking area. These processes have characteristics of both "point fire" and "area fire" models. A certain probabilistic model of an attack process is postulated; from it, exact and approximate expressions for expected numbers of targets killed are derived for a number of different sets of variations in the assumptions underlying the model. The paper explores in detail the relationship between these expressions and several previously-developed attrition equations in which targets were assumed not to be located on parking areas. The paper also provides rigorous mathematical justification for three equations that have been used in several combat simulations to compute attrition to aircraft on the ground. These equations are shown to be different special cases of the general model.				
20. DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS		21. ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a. NAME OF RESPONSIBLE INDIVIDUAL		22b. TELEPHONE (Include Area Code)	22c. OFFICE SYMBOL	

IDA PAPER P-2144

## ATTRITION PROCESSES WITH PARKING AREAS

Eleanor L. Schwartz

September 1988



INSTITUTE FOR DEFENSE ANALYSES

IDA Independent Research Program

## PREFACE

This paper was prepared under the Institute for Defense Analyses' Central Research Program. It is an expansion and mathematical motivation of some concepts incorporated into several IDA-developed combat simulation models.

The author is grateful to Dr. Lowell Bruce Anderson, Dr. Peter Brooks, and Dr. Alan Rolfe for their review of this paper.

Accession For	
NTIS GRA&I	<input checked="" type="checkbox"/>
DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Avail and/or	
Dist	Special

A-1

## ABSTRACT

This paper examines a class of combat processes in which targets can be located on "parking areas," so that an attack on a target can kill other targets on the same parking area. These processes have characteristics of both "point fire" and "area fire" models. A certain probabilistic model of an attack process is postulated; from it, exact and approximate expressions for expected numbers of targets killed are derived for a number of different sets of variations in the assumptions underlying the model. The paper explores in detail the relationship between these expressions and several previously-developed attrition equations in which targets were assumed not to be located on parking areas. The paper also provides rigorous mathematical justification for three equations that have been used in several combat simulations to compute attrition to aircraft on the ground. These equations are shown to be different special cases of the general model.

## CONTENTS

PREFACE .....	iii
ABSTRACT.....	v
A. INTRODUCTION.....	1
B. ATTRITION PROCESSES WITH PARKING AREAS AND UNIFORM ALLOCATION OF FIRE .....	2
1. Basic Structure of the Processes .....	2
a. Terminology and Assumptions .....	2
b. Probabilistic Arguments.....	5
2. The Case Where Nonzero Detection Probabilities Are a Function of Searcher Type Only .....	7
a. Derivation of Attrition Equation.....	8
b. A Further Special Case--All Detection Probabilities a Function of Searcher Type Only.....	10
3. Two Specific Attrition Processes with Parking Areas and Uniform Allocation of Fire.....	10
a. ATRTAB Option 3.....	11
b. ATRTAB Option 2 .....	12
4. Toward a General Attrition Equation .....	15
a. Use of Poisson Approximations.....	15
b. An Approximate General Attrition Equation of Binomial Form.....	18
C. PARKING AREAS AND STRICT PRIORITY ALLOCATION OF FIRE.....	20
1. Introduction and Assumptions.....	20
2. Derivation of Attrition Equation.....	21
3. A Specific Case--ATRTAB Option 1.....	23
D. PROCESSES IN WHICH A SEARCHER DETECTS PARKING AREAS.....	24
1. Terminology, Assumptions, and Proof Elements .....	25
2. Equations for the Uniform Allocation of Fire Cases .....	27
a. Nonzero Detection Probabilities a Function of Searcher Type Only.....	27
b. All Detection Probabilities a Function of Searcher Type Only.....	28
c. Approximate Attrition Equations With General Detection Probabilities.....	28

3. Equation for the Strict Priority Allocation of Fire Case .....	30
4. Reduction to the "No-Parking-Areas" Case.....	31
<b>E. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK .....</b>	<b>32</b>
<b>REFERENCES.....</b>	<b>R-1</b>

## A. INTRODUCTION

In aggregated, deterministic, large-scale models of combat (as opposed to fine-grained or Monte Carlo models), attrition to resources is often computed, implicitly or explicitly, by means of attrition equations. The results of such models are, in general, sensitive to the particular equations used. Accordingly, the study of attrition equations is an important aspect of large-scale modeling. Moreover, the realism of such models is increased if the equations used to compute attrition in a certain combat interaction take into account appropriate specific features of that interaction. If an attrition equation is derived mathematically from more fundamental assumptions about combat, and if those fundamental assumptions consider some appropriate features of combat, then the usefulness of such an equation will be enhanced, its properties and data requirements can be better understood, and the mathematical aspects of its suitability (or lack thereof) can be verified.

To take a specific example, an equation used to compute attrition to aircraft on an airbase (while on the ground) could consider that the aircraft might be parked in clusters ("aprons") and that an attack on an aircraft might kill some other aircraft in the same cluster.

Accordingly, this paper examines several stochastic attrition processes where targets can be located on "parking areas," in such a manner that an attack on a specific target might be able to kill other targets on the same parking area. Several attrition equations are derived; these represent exact or approximate formulas for the expected number of targets killed, under varying assumptions about the combat processes. Most of the attrition equations derived are of the "binomial" form; all can be considered discrete-time analogs of Lanchester equations, as discussed in References [5], [6], and [7].

One motivation for this paper is that three specific attrition equations with parking areas have been used in several combat simulation models to compute attrition to aircraft on airbases (caused by enemy aircraft). These equations, which were originally derived by L.B. Anderson, have been used in the IDATAM [3], NAVMOD [4], and OPTSA [2] models, among others. They differ in their assumptions on the choice of targets an attacker attacks and the specific targets that can be killed by an attack. (For any given run of the model, any one of these equations can be selected, by input, to calculate attrition.) In NAVMOD and IDATAM, these equations are used in a subroutine named ATRTAB (mnemonic for "attrition at airbase"), thus they will be referred to here as the "ATRTAB options." This paper develops generalizations of these equations and proves that they

correctly give the formulas for the expected numbers of targets killed under three different variations of the set of combat assumptions.<sup>1</sup> This paper also extends several previously-developed attrition equations to treat the case of parking areas.

The combat processes examined here are divided into three groups, based on the protocol by which a searcher chooses a target to attack. Section B looks at processes with "uniform allocation of fire," in which each searcher selects a target to attack uniformly from the targets it has detected. Several special cases are developed, including the second and third ATRTAB equations and generalizations of attrition equations discussed in References [9] and [11]. Section C considers a process with "strict priority allocation of fire," in which a searcher attacks the highest priority target it has detected (where priority is based on type of target). This work extends the combat process of Reference [8] to include parking areas, and the first ATRTAB equation falls out as a special case. Section D examines some combat processes in which a searcher detects whole parking areas, rather than individual targets.

This paper subsumes and supersedes Reference [13], which discussed some aspects of the combat process with parking areas and uniform allocation of fire.

## B. ATTRITION PROCESSES WITH PARKING AREAS AND UNIFORM ALLOCATION OF FIRE

### 1. Basic Structure of the Processes

#### a. Terminology and Assumptions

The terminology introduced in this subsection and Subsection B.1.b will be used throughout the paper. Suppose that there are  $m$  types of searchers and  $n$  types of targets. Let  $i=1,\dots,m$  index searcher types and  $j=1,\dots,n$ , target types. Let  $\{1,\dots,n\}$  be partitioned into  $A$  sets  $U_1,\dots,U_A$ ; think of the index  $a=1,\dots,A$  as denoting "type of parking area." The event that  $j \in U_a$  (for some specific pair of  $j$  and  $a$ ) should be interpreted as "targets of

---

<sup>1</sup> See Reference [4], Chapter IV, Section C.2.c, for more information on Subroutine ATRTAB. The current paper generalizes the central attrition calculations of Subroutine ATRTAB, but the subroutine also performs certain additional combat-related calculations not directly germane to this paper. It assumes only two types of targets: open (nonsHELTERED) aircraft and aircraft shelters, but some of the shelters might be occupied, and the subroutine makes additional calculations to determine a number of sheltered aircraft killed. Also, NAVMOD, IDATAM, and OPTSA model airbase attack as being directed against a set of identical "typical" airbases. Subroutine ATRTAB computes attrition for one such typical airbase; these results are then multiplied by the number of such airbases to compute overall attrition. See References [3] and [4] for details.

type  $j$  are located on parking areas of type  $a$ ." Since the sets  $U_a$  form a partition, each given type of target can be located on exactly one type of parking area. For each target type  $j$ , let  $a(j)$  denote this parking area type; the distinction between the subscript  $a$  and the function  $a(j)$  should be clear from context. Suppose that  $T_j$  targets of type  $j$  ( $j=1, \dots, n$ ) and  $M_a$  parking areas of type  $a$  ( $a=1, \dots, A$ ) are present. Then each type- $a$  parking area is considered as having  $t_j = T_j/M_a$  targets located on it, for each  $j$  in  $U_a$  (and as having no type- $j$  targets located on it for all  $j$  not in  $U_a$ ). For each value of  $j$ ,  $M_{a(j)}$  denotes the number of parking areas that can accommodate type- $j$  targets, and  $U_{a(j)}$  denotes the particular set (out of  $U_1, \dots, U_A$ ) that contains  $j$ .<sup>1</sup>

Suppose that there are  $S_i$  searchers of type  $i$  present ( $i=1, \dots, m$ ). All searchers of any given searcher type and all targets of any given target type are assumed to be identical, and, as indicated above, all parking areas of any given parking area type are assumed to contain identical complements of targets. Consider an attrition process that proceeds according to the following assumptions.

- (1) At a fixed time, all targets become vulnerable to detection and attack.
- (2) Any particular searcher of type  $i$  detects any particular target of type  $j$  with probability  $d_{ij}$ .
- (3) Detections of different targets by a given searcher are mutually independent events.
- (4) The detection and attack processes of different searchers are mutually independent.
- (5) Of the targets it has detected, each searcher chooses one target according to a uniform distribution, and makes an (one) attack on the parking area containing that target. (A searcher that makes no detections makes no attack.)
- (6) If an attack by a type- $i$  searcher is made on a given parking area, then each type- $j$  target located on that parking area is killed with probability  $k_{ij}$ . The effects of different attacks on the same parking area are independent.

---

<sup>1</sup> The derivations in this paper are valid only if all  $S_i$  are nonnegative integers, all  $M_a$  are strictly positive integers, and  $T_j$  is a nonnegative integer multiple of  $M_{a(j)}$  for each  $j$  (so that all  $t_j$  are nonnegative integers). The resulting equations, however, can be evaluated with any nonnegative (real)  $S_i$  and  $T_j$  and real  $M_a \geq 1$ , but are considered reasonable only if  $M_{a(j)} \leq T_j$  for each  $j$  where  $T_j$  is nonzero. More research is needed to develop consistent and reasonable attrition equations that address the case in which this condition does not hold. Currently, Subroutine ATRTAB contains certain computations of a heuristic nature to treat this case. (The iterative use of attrition equations in a combat model can yield noninteger numbers of combatants.)

It is desired to find the expected number of type- $j$  targets killed, for each  $j$  ( $j=1, \dots, n$ ). The possibility that two or more searchers kill (i.e., lethally attack) the same target is explicitly considered.

Assumption (5) can be called the "uniform allocation of fire" rule for target choice. An assumption of this sort has been made in much of the previous work on binomial attrition processes (e.g., References [5], [6], [7], and [10]) and stands in contrast to the "strict priority allocation of fire" assumption of Reference [8] and Section C of the current paper and the "weighted allocation of fire" rule discussed in References [1] and [12].

Two points about Assumption (6) should be noted. First, it is possible for a target to be killed without having been detected--if some other target on the same parking area has been detected and that parking area has been attacked. Indeed, in some cases, it is possible for targets of type  $j$  to not be directly detectable--i.e.,  $d_{ij} = 0$  for that  $j$  and all  $i$ --yet still suffer attrition. Second, to compute the expected number of targets killed, no knowledge is necessary of the joint distribution of the targets killed on a parking area, given attack--the marginal probabilities of kill given attack  $k_{ij}$  suffice.

This combat process has elements of both "point fire" and "area fire" models. It is like point fire in that a searcher must make a detection in order to attack, but is like area fire in that one attack by one searcher can kill several targets, including targets not explicitly detected by that searcher.

At the outset, we emphasize that this paper does not derive an exact formula for the expected number of targets killed in this process. Note that in Assumption (2), the detection probabilities are a function of both searcher type and target type. To date, it has not been possible to derive succinct, exact expressions for expected numbers of targets killed in attrition processes where this condition holds and a uniform allocation of fire assumption is made (see References [5] and [9]). Section B.4 below develops several approximate formulas for the expected number of targets killed in the general process; Sections B.2 and B.3 derive exact closed-form expressions for the expected number of targets killed in several special cases. Most of these special cases involve some set of restrictions on the detection probabilities  $d_{ij}$ .

Suppose that each parking area contains exactly one target. Then Assumption (6) collapses into the regular assumption that an attack made on a target can kill only that target; i.e., the assumptions stated above become equivalent to those of the basic "no-parking-areas" attrition process of Chapter III of Reference [5]. No closed form expression for the

expected number of targets killed was derived for that process, and the expression that was derived is algebraically intractable. It is thus not surprising that no succinct closed form expression suggests itself for the expected number of targets killed in the general process with parking areas and uniform allocation of fire.

On the other hand, several approximate and special-case formulas have been developed in the no-parking-areas case for the expected number of targets killed (see References [7], [9], and [1]). In this paper, these formulas are extended to incorporate the feature of parking areas. Suppose that the parameters satisfy the assumptions:

$$\left. \begin{array}{ll} A=n \\ a(j)=j & \text{for } j=1, \dots, n, \\ U_{a(j)}=\{j\} & \text{for } j=1, \dots, n, \text{ and} \\ M_{a(j)}=T_j & \text{for } j=1, \dots, n. \end{array} \right\} \quad (B.1.1)$$

Then each parking area indeed does contain exactly one target,<sup>1</sup> and one would expect a formula for expected targets killed in a "parking areas" case to reduce to the corresponding "no-parking-areas" formula. This in fact does occur, and will be pointed out as appropriate.

### b. Probabilistic Arguments

The definitions and probabilistic arguments presented in this section both indicate the difficulties of finding a general expression for the expected number of targets killed and can be used to derive many of the approximate and special-case expressions.

Let  $T_j^K$  denote the expected number of type-j targets killed. We first state a basic lemma, which is true by Assumptions (4) and (6) and standard methods (see Reference [11]).

#### Lemma 1:

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m (1-H_{ij})^{S_i} \right], \quad (B.1.2)$$

where  $H_{ij}$  denotes the probability that a specific type-j target is killed by a specific type-i searcher.

---

<sup>1</sup> More generally, if the mapping  $a$  is any permutation of  $\{1, \dots, n\}$ , and  $U_{a(j)} = \{j\}$  and  $M_{a(j)} = T_j$  for each  $j=1, \dots, n$ , then each parking area contains exactly one target.

Let  $\sigma$  denote a specific searcher of type  $i$ , and  $\tau$ , a specific target of type  $j$ . Suppose that target  $\tau$  is located on parking area  $\alpha$  (thus  $\alpha$  is of type  $a(j)$ ). Define the following events:

$K-\sigma$  kills  $\tau$  (the probability  $P(K)$  is equal to  $H_{ij}$  as defined above),

$F_\alpha-\sigma$  attacks parking area  $\alpha$ ,

$G_{a(j)}$   $\sigma$  attacks a parking area of type  $a$ , i.e.,  $\sigma$  makes an attack and the parking area  $\sigma$  attacks is of type  $a$  (defined for  $a=1, \dots, A$ ), and

$D-\sigma$  detects at least one target (regardless of type).

From Assumption (6) it is clear that  $P(K|F_\alpha) = k_{ij}$  and, of course  $P(K|\bar{F}_\alpha) = 0$  thus

$$H_{ij} \equiv P(K) = k_{ij} P(F_\alpha). \quad (B.1.3)$$

From Assumptions (2) and (3),

$$P(D) = 1 - \prod_{r=1}^n (1-d_{ir})^{T_r}. \quad (B.1.4)$$

By Assumption (5), searcher  $\sigma$  will attack no more than one parking area, and will attack a parking area if and only if it detects at least one target. Thus different events  $G_a$  are disjoint and

$$\bigcup_{a=1}^A G_a = D,$$

and thus

$$\sum_{a=1}^A P(G_a) = 1 - \prod_{r=1}^n (1-d_{ir})^{T_r}. \quad (B.1.5)$$

All parking areas of a given type are assumed to be identical, and at the outset, all targets of any given target type are equally detectable by searcher  $\sigma$ . Thus

$$P(F_\alpha|G_{a(j)}) = 1/M_{a(j)}. \quad (B.1.6)$$

(Of course,  $P(F_\alpha|G_a) = 0$  for  $a \neq a(j)$ .) If the probabilities  $P(G_a)$  could be obtained, then  $P(K)$ , i.e.,  $H_{ij}$ , could be computed as

$$H_{ij} \equiv P(K) = k_{ij} P(G_{a(j)})/M_{a(j)}, \quad (B.1.7)$$

and the expected attrition would follow forthwith from Lemma 1.

The probabilities  $P(G_a)$  are not evident in the general case, but can be derived in several special cases.

Equation (B.1.3) reflects the fact that searcher  $\sigma$  kills target  $\tau$  with (conditional) probability  $k_{ij}$  if  $\sigma$  attacks  $\tau$ 's parking area (parking area  $\alpha$ ), whether or not  $\tau$  is the "primary target" of  $\sigma$ 's attack (in the sense that  $\sigma$  "chooses"  $\tau$  in Assumption (5)). The idea of "primary target" can be used, however, to develop an alternative formula for  $P(F_\alpha)$ . The event that searcher  $\sigma$  attacks parking area  $\alpha$  equals the event that some target on parking area  $\alpha$  is the primary target of  $\sigma$ 's attack. Any given searcher will have at most one "primary target" in any realization of the combat process. All targets of a given type are assumed to be identical, and parking area  $\alpha$  contains  $t_r$  (which equals  $T_r/M_{a(j)}$ ) targets of type  $r$ , for each  $r$  in  $U_{a(j)}$ . All in all, then,

$$P(F_\alpha) = \sum_{r \in U_{a(j)}} t_r c_{ir} , \quad (B.1.8)$$

where  $c_{ir}$  is the probability that a specific type- $r$  target is the primary target of searcher  $\sigma$ 's attack (recall that  $\sigma$  is of type  $i$ ). (Strictly speaking, if  $t_r = 0$ ,  $c_{ir}$  is undefined, but in this case interpret the product  $t_r c_{ir}$  as zero.) In some special cases, as indicated below, it is possible and expeditious to compute the  $c_{ir}$ . Applying (B.1.8), (B.1.3) and Lemma 1 yields  $T_j^K$ .

## 2. The Case Where Nonzero Detection Probabilities Are a Function of Searcher Type Only

In this section it is assumed that the one-on-one detection probabilities  $d_{ij}$  are either zero or dependent only on the type ( $i$ ) of searcher. More formally, the following assumption is made.

$$\left. \begin{array}{l} \text{For each searcher type } i \text{ } (i=1,\dots,m) \text{ there exists} \\ \text{a value } \bar{d}_i \in [0,1] \text{ and a subset } J_i \text{ of the set} \\ \text{of target types } \{1,\dots,n\} \text{ such that} \\ \text{--for all } j \in J_i, d_{ij} = \bar{d}_i, \text{ and} \\ \text{--for all } j \notin J_i, d_{ij} = 0. \end{array} \right\} \quad (B.2.1)$$

Some of all of the sets  $J_i$  could be empty or equal to the whole space  $\{1,\dots,n\}$ . (The formulas below make sense if  $\bar{d}_i = 0$  for some  $i$ , but the "natural" interpretation is that all  $\bar{d}_i$  are strictly positive.)

In this case, an exact and relatively concise expression for the expected number of targets killed can be derived, as will now be shown.

a. Derivation of Attrition Equation

Again, consider a specific searcher  $\sigma$ , of type  $i$ , and let  $D$  denote the event that  $\sigma$  detects at least one target. From Assumptions (2) and (3) and condition (B.2.1) it is clear that

$$P(D) = 1 - (1 - \bar{d}_i)^{\bar{V}_i}, \quad (B.2.2)$$

where (the integer)  $\bar{V}_i$  is defined by

$$\bar{V}_i = \sum_{r \in J_i} T_r. \quad (B.2.3)$$

One should think of  $\bar{V}_i$  as being the maximum number of targets that  $\sigma$  can detect. For now, assume that  $\bar{d}_i$  and  $\bar{V}_i$  are both strictly greater than zero. At the outset, all of these  $\bar{V}_i$  targets are equally detectable by searcher  $\sigma$  (because of condition (B.2.1)). By Assumption (5), the choice of primary target is made uniformly from the detected targets. Then by symmetry, the probabilities that any specific one of the  $\bar{V}_i$  targets detectable by  $\sigma$  becomes  $\sigma$ 's primary target are equal for all of the  $\bar{V}_i$  targets. Also by Assumption (5), searcher  $\sigma$  will choose exactly one primary target whenever  $\sigma$  detects at least one target. Thus the probability that any specific target detectable by  $\sigma$  is the primary target of  $\sigma$  is equal to

$$P(D)/\bar{V}_i. \quad (B.2.4)$$

(This symmetry argument is similar to that of Reference [12]; expression (B.2.4) could also be derived by reasoning of the type given in Reference [11].)

Now consider a specific target  $\tau$ , of type  $j$ , located on parking area  $\alpha$  (which is thus of type  $a(j)$ ), and let the events  $K$  and  $F_\alpha$  and the probability  $H_{ij} = P(K)$  be as described in Section B.1.b. To compute  $H_{ij}$ , first note that if any or all of the conditions  $\bar{d}_i = 0$ ,  $\bar{V}_i = 0$ , and/or  $J_i = \emptyset$  hold ( $\emptyset$  denotes the null set), searchers of type  $i$  can detect no targets at all, and  $H_{ij} = 0$  (and  $H_{ir} = 0$  for all target types  $r$ ). Otherwise,  $H_{ij}$  can be computed from equations (B.1.3) and B.1.8); i.e.,  $\tau$  can be killed by  $\sigma$  precisely when one of the targets on parking area  $\alpha$  is the primary target of  $\sigma$ 's attack. Parking area  $\alpha$  contains  $t_r (=T_r/M_{a(j)})$  targets of type  $r$ , for each  $r \in U_{a(j)}$ . Consider a target type  $r \in U_{a(j)}$  such that  $T_r > 0$ , i.e., targets of type  $r$  are indeed present. If  $r \notin J_i$ , then searcher  $\sigma$  cannot detect type- $r$

targets, and thus a target of type  $r$  will never be the primary target of  $\sigma$ 's attack. If  $r \in J_i$ , then any particular type- $r$  target will be the primary target of  $\sigma$ 's attack with probability  $P(D)/\bar{V}_i$  (expression (B.2.4)). Substituting into equation (B.1.8) yields

$$P(F_\alpha) = \sum_{r \in U_{a(j)} \cap J_i} t_r P(D)/\bar{V}_i. \quad (B.2.5)$$

Combining the above results and applying (B.1.3) and Lemma 1 yields

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m (1 - H_{ij}) \right]^{S_i}, \quad (B.2.6)$$

where

$$H_{ij} = \begin{cases} 0, & \bar{V}_i = 0, \\ \frac{k_{ij}}{M_{a(j)}} \left( \sum_{r \in U_{a(j)} \cap J_i} T_r \bar{V}_i \right) \left[ 1 - (1 - \bar{d}_i) \bar{V}_i \right], & \text{otherwise,} \end{cases} \quad (B.2.7)$$

where sums over the null set are considered to be zero.

Note that it is possible that for certain target types  $j$ , the detection probability  $d_{ij}$  might equal zero for all searcher types  $i$  (e.g., for all  $j \notin J_i$ ), yet  $T_j^K$  as computed by (B.2.6) and (B.2.7) might be strictly greater than zero. This can be seen formally from equation (B.2.7): even if  $j \notin J_i$ ,  $U_{a(j)} \cap J_i$  could be nonempty and  $\bar{d}_i$  could be nonzero.

That is, type- $j$  targets, though not directly detectable, could be located on the same parking areas as detectable targets, and thus receive fire from an attacker, even though they could not be primary targets of an attacker.

If the parameters satisfy the conditions (B.1.1), i.e., each parking area contains exactly one target, it can be verified that  $T_j^K$  as computed by (B.2.6) and (B.2.7) reduces to equation (21) of Reference [9] (which is also equation (12) of Reference [11]). In this case, the property described in the preceding paragraph cannot occur, for under (B.1.1),  $U_{a(j)} = \{j\}$  and thus in the expression for  $H_{ij}$  in (B.2.7) either the indicated sum or the term in brackets (or both) will be zero unless  $d_{ij} > 0$ .

b. A Further Special Case--All Detection Probabilities a Function of Searcher Type Only

Now consider the case where the detection probabilities satisfy a restricted version of the condition (B.2.1), such that for each searcher type  $i$ ,  $J_i$  is the full set of target types  $\{1, \dots, n\}$ . Then, for each  $i$ ,  $d_{ij} = \bar{d}_i$  for all  $j$ . (For some  $i$ ,  $\bar{d}_i$  may be zero; this implies that searchers of type  $i$  are completely ineffective.) In this case, the attrition equation given by (B.2.6) and (B.2.7) reduces to

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{TM_{a(j)}} \left( \sum_{r \in U_{a(j)}} T_r \right) \left[ 1 - (1 - \bar{d}_i)^T \right] \right)^{S_i} \right] \quad (B.2.8)$$

where  $T = \sum_{r=1}^n T_r$  denotes the total number of targets.

As in the more general case of the preceding subsection, it is possible that for some  $j$ ,  $d_{ij} = 0$  for all  $i$  yet  $T_j^K > 0$ . If there is one target per parking area, i.e., the conditions (B.1.1) hold, then equation (B.2.8) reduces to the "basic heterogeneous binomial attrition equation," which has been used in a number of combat models (including References [2], [3], and [4]), and is derived and discussed in References [5], [6], [11], and [12].

3. Two Specific Attrition Processes With Parking Areas and Uniform Allocation of Fire

This section derives two attrition equations that are special cases of the general process described in Section B.1. The notation of Section B.1 is used throughout. Further restrictions on these special cases yield the second and third attrition equations available in (Subroutine ATRTAB of) the NAVMOD and IDATAM models to compute attrition to targets on an airbase caused by enemy attacking aircraft. Here, however, the special cases themselves will be referred to as "ATRTAB Option 2" and "ATRTAB Option 3," even though the equations are somewhat more general than the equations in the ATRTAB computer code.

The detection probabilities of these special cases need not satisfy the restrictions (B.2.1) of Section B.2. Even though there does not appear to be a simple attrition equation for the general process, in these special cases enough extra information is available to compute the probabilities  $P(G_a)$  of equation (B.1.7) and thus to develop a formula for the expected number of targets killed.

For ease in presentation, the special cases are discussed here in a different order than they appear in Subroutine ATRTAB. The first special case discussed here corresponds to Option 3 of ATRTAB, and the second to Option 2. The first option of Subroutine ATRTAB involves priority, not uniform, allocation of fire, and is discussed in Section C.3, below.

a. ATRTAB Option 3

This special case assumes that there is just one type of parking area, i.e.,  $A=1$ . Then the sum

$$\sum_{a=1}^A P(G_a)$$

which equals  $P(D)$ , consists of a single term  $P(G_1)$ , and  $a(j) = 1$  for all  $j$ . Substituting in equation (B.1.7) yields, for all  $i$  and  $j$ ,

$$H_{ij} \equiv P(K) = k_{ij} P(D)/M_1. \quad (B.3.1)$$

Using expression (B.1.4) for  $P(D)$  and substituting  $H_{ij}$  as above into equation (B.1.2) yields, for this special case,

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{M_1} \left[ 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \right] \right)^{S_i} \right]. \quad (B.3.2)$$

Note that in equation (B.3.2), the detection probabilities appear only in a product, and if the term

$$P(D) = 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \quad (B.3.3)$$

is nonzero, even if many of the  $d_{ir}$  are zero, then  $T_j^K$  can be nonzero for each  $j$ . That is, if a type- $i$  searcher can detect at least one type of target, it can potentially kill a target of any type. For if there is only one type of parking area, then by the parameter assumptions, each parking area contains targets of all types.

Suppose that the conditions (B.1.1) hold, i.e., each parking area contains exactly one target, and also the assumption  $A=1$  of ATRTAB Option 3 holds. Then  $n=1$ --i.e., there is only one type of target--and equation (B.3.2) reduces to a special case of the "basic heterogeneous binomial attrition equation" (equation (3.14) of Reference [6]; see also the references listed in Section B.2.b, above) where there is only one type of target.

### b. ATRTAB Option 2

This case arises if the detection probabilities  $d_{ij}$  are such that searchers of any given type can attack only one type of parking area. In general, one can consider, for each  $i$ , the set

$$J_i = \{j \mid d_{ij} > 0\}$$

(and  $j$  integer,  $1 \leq j \leq n$ ) of target types that a type- $i$  searcher can potentially detect. (This notation is slightly different from that of Section B.2.) ATRTAB Option 2 assumes that the detection probabilities  $d_{ij}$  satisfy the following condition:

$$\left. \begin{array}{l} \text{For each searcher type } i, \text{ there exists some} \\ \text{integer } b(i) \text{ (between 1 and } A, \text{ inclusive) such} \\ \text{that } J_i \subset U_{b(i)}. \end{array} \right\} \quad (B.3.4)$$

Consider the example  $m=2$ ,  $n=6$ ,  $A=3$ ,  $U_1=\{1\}$ ,  $U_2=\{2,3,4\}$ ,  $U_3=\{5,6\}$ ,  $d_{12}$ ,  $d_{14}$ , and  $d_{25}$  are strictly greater than zero, and all other  $d_{ij}$  equal zero. Then  $J_1=\{2,4\}$ , which is a subset of  $U_2$ , and  $J_2=\{5\}$ , which is a subset of  $U_3$ . Thus the detection probabilities of this example satisfy condition (B.3.4), with  $b(1)=2$  and  $b(2)=3$ .<sup>1,2</sup>

---

<sup>1</sup> Condition (B.3.4) also holds for the special case in which each  $J_i$  has one element, i.e., each type of searcher can potentially detect only one type of target.

<sup>2</sup> It is evident that if there is only one type of parking area ( $A=1$ ) then the detection probabilities satisfy condition (B.3.4) with  $b(i)=1$  for all  $i$ , since  $U_1$  is the set of all target types. Thus ATRTAB Option 3 is a special case of ATRTAB Option 2. Because of its additional simplicity, ATRTAB Option 3 has been presented in a separate section. It can be verified that the formula (B.3.9) for expected attrition in ATRTAB Option 2 reduces to equation (B.3.2) if  $A=1$ .

The sets  $U_a$  form a partition, thus if condition (B.3.4) is satisfied, then  $b(i)$  is unique for each  $i$ . This means that searchers of a given type can only detect--and thus (by Assumption (5)) attack--targets located on one type of parking area. The reverse need not be true: a given type of parking area might be attackable by several types of searchers--or no searchers at all. (In this latter case, all the targets located on such a parking area survive, as occurs with type-1 targets in the above example).

As always, the attrition to type- $j$  targets can be computed by first computing, for each searcher type  $i$ ,  $H_{ij}$  or  $P(K)$ , the probability that a specific type- $i$  searcher kills a specific type- $j$  target, and substituting in equation (B.1.2). Type- $j$  targets are located on parking areas of type  $a(j)$  and in ATRTAB Option 2, type- $i$  searchers can only attack parking areas of type  $b(i)$ . As before, let  $\sigma$  be some specific type- $i$  searcher and  $\tau$ , some specific type- $j$  target. The probabilities  $P(G_a)$  that  $\sigma$  attacks some parking area of type  $a$  are zero except for  $a = b(i)$ . It is still true that

$$\sum_{a=1}^A P(G_a) = P(D) = 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r},$$

thus

$$P(G_a) = \begin{cases} P(D) & a = b(i), \\ 0 & \text{otherwise.} \end{cases}$$

In particular

$$P(G_{a(j)}) = \begin{cases} P(D) & a(j) = b(i), \\ 0 & \text{otherwise;} \end{cases} \quad (B.3.5)$$

substituting into equation (B.1.7) yields

$$H_{ij} \equiv P(K) = \begin{cases} k_{ij} [1 - \prod_{r=1}^n (1 - d_{ir})^{T_r}] / M_{a(j)} & b(i) = a(j) \\ 0 & \text{otherwise.} \end{cases} \quad (B.3.6)$$

The definition of  $b(i)$  implies that

$$b(i) = a(j) \quad \text{iff} \quad J_i \subset U_{a(j)}. \quad (B.3.7)$$

For each  $j$ , define the subset of  $\{1, \dots, m\}$

$$C_j = \{i \mid J_i \subset U_{a(j)}\}. \quad (B.3.8)$$

Given that condition (B.3.4) holds,  $C_j$  is the set of types of searchers that can attack type-a(j) parking areas. It is possible that  $C_j$  is the null set for some  $j$ --this occurs if all the types of targets located on type-a(j) parking areas are undetectable by searchers of any type. Substituting the  $H_{ij}$  as defined above into equation (B.1.2) and simplifying yields, for each  $j$  in turn,

$$T_j^K = T_j \left[ 1 - \prod_{i \in C_j} \left( 1 - \frac{k_{ij}}{M_{a(j)}} \left[ 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \right] \right)^{S_i} \right], \quad (B.3.9)$$

where a product taken over the null set is interpreted as unity.

It is interesting to note that if condition (B.3.4) holds, then (for each  $j$ ) the set  $C_j$  defined above always contains the set

$$I_j = \{i \mid d_{ij} > 0\} = \{i \mid j \in J_i\} \quad (B.3.10)$$

but is not necessarily equal to  $I_j$ . ( $I_j$  is the set of types of searchers that can detect type-j targets.) In the example mentioned earlier,  $I_3$  is the null set, but  $C_3 = \{1\}$ --and note that type-3 targets, by being co-located with type-2 and type-4 targets, are potentially vulnerable to type-1 searchers, though not directly detectable by them.

The implementation of ATRTAB Option 2 in the combat models of References [2], [3], and [4] at first glance does not appear similar to the description above. In that implementation, the attacker's sorties are preallocated between attacking open aircraft only or aircraft shelters only. A sortie preallocated to attack open aircraft only is not allowed to attack (or kill) a shelter, even if it detects some shelters but no open aircraft, and vice versa. In the process of Section B.1 (of the current paper), Assumption (5) implies that a searcher is capable of potentially attacking any target it detects, regardless of type. If different types of targets are located on different types of parking areas, however, preallocation can be modeled in this process by considering searchers preallocated to different target types as different types (i) of searchers and by setting, for each  $i$ ,  $d_{ij} = 0$  for all target types  $j$  except the one type  $j(i)$  to which type- $i$  searchers are preallocated. Then only targets of type  $j(i)$  can be the primary targets of a type- $i$  searcher's attack, and no targets of other types can be incidentally killed by such a searcher. Furthermore, each set  $J_i$  then consists of the single element  $j(i)$ , thus condition (B.3.4) is satisfied and equation (B.3.9), i.e., ATRTAB Option 2 as derived in the current paper, can be used to compute attrition. In the implementation of this option in the models of References [2], [3], and [4], open aircraft and aircraft shelters

are indeed assumed to be located on different types of parking areas (and in addition, the number of "parking areas for shelters" equals the number of shelters, so that an attack on a shelter can kill only that shelter). (It is clear that several generalizations of this structure might be possible--e.g., requiring the preallocation of primary target types but allowing incidental kills of targets of other types.)

#### **4. Toward a General Attrition Equation**

All the attrition equations presented so far have been exact expressions for the expected number of targets killed, i.e., they can be rigorously derived from the assumptions of the combat process. In contrast, this section presents and gives justification for three attrition equations which are approximate expressions for the expected number of targets killed. These expressions accept general  $d_{ij}$  and have concise forms. Subsection a extends some concepts of Reference [9] to the parking areas case, developing two equations that contain exponential terms. Subsection b presents an equation of "binomial" form that reduces to the appropriate exact expression in each of the special cases of Sections B.2, B.3.a, and B.3.b, above.

##### **a. Use of Poisson Approximations**

This section develops an approximate formula for the probability  $c_{ir}$  that a specific type- $r$  target (target  $\rho$ ) is the primary target of a specific type- $i$  searcher (searcher  $\sigma$ ). Equations (B.1.8), (B.1.3), and (B.1.2) (Lemma 1) can then be applied to yield an approximate expression for the expected number of type- $j$  targets killed,  $T_j^K$ , for each  $j$  in turn.

The following arguments are essentially identical to those of Reference [9], with different notation. Define the following random variables:

$L_r$  = number of targets of type  $r$ , other than target  $\rho$ , that searcher  $\sigma$  detects,

$L_s$  = number of targets of type  $s$  that searcher  $\sigma$  detects (each subscript  $s \neq r$  defines a different random variable), and

$$L = \sum_{s=1}^n L_s = \text{total number of targets, other than target } \rho, \text{ that searcher } \sigma \text{ detects.}$$

By Assumptions (2) and (3),

- $L_r$  follows a binomial distribution with parameters  $T_r - 1$  and  $d_{ir}$ ,
- $L_s$  (for each  $s \neq r$ ) follows a binomial distribution with parameters  $T_s$  and  $d_{is}$ , and
- All random variables  $L_r$  and  $L_s$  are mutually independent.

Also, searcher  $\sigma$  detects target  $\rho$  with probability  $d_{ir}$ , and by Assumption (5)

$$c_{ir} = \sum_{l=0}^{T-1} \frac{1}{l+1} d_{ir} P(L=l), \quad (B.4.1)$$

where  $T$  denotes the total number of targets.<sup>1</sup> Unfortunately, since the probabilities  $d_{is}$  are not necessarily the same for different  $s$ , the probability distribution of  $L$  does not have a simple form. Use of the Poisson approximation to the binomial, however, yields the following results:

- $L_r$  is approximately distributed Poisson with mean  $d_{ir}(T_r - 1)$ ,
- $L_s$  is approximately distributed Poisson with mean  $d_{is}T_s$  (for each  $s=1, \dots, n$ ;  $s \neq r$ ), and
- $L$  is approximately distributed Poisson with mean

$$\mu_{ir} = d_{ir}(T_r - 1) + \sum_{s \neq r} d_{is} T_s. \quad (B.4.2)$$

Therefore

$$c_{ir} \approx \sum_{l=0}^{\infty} \frac{1}{l+1} d_{ir} \frac{(\mu_{ir})^l e^{-\mu_{ir}}}{l!} = \frac{d_{ir}}{\mu_{ir}} (1 - e^{-\mu_{ir}}). \quad (B.4.3)$$

(It is conceivable that  $\mu_{ir}=0$ . Given that all parameters are nonnegative and that the numbers of searchers and targets are integers, this implies that target  $\rho$  is the only type- $r$  target present and that any other targets present are undetectable by searcher  $\sigma$ . Thus, if  $\sigma$  detects  $\rho$ ,  $\rho$  will be the primary target of  $\sigma$ 's attack, thus  $c_{ir}=d_{ir}$ . This is consistent with the interpretation of  $(1-e^{-\mu_{ir}})/\mu_{ir} = 1$  at  $\mu_{ir} = 0$  via L'Hopital's rule.)

Applying equations (B.1.8) and (B.1.3) and Lemma 1 yields

---

<sup>1</sup> Throughout this paper, the symbol "l" is to be read as a lower case "ell."

$$T_j^K \approx T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{M_{a(j)}} \sum_{r \in U_{a(j)}} \frac{d_{ir} T_r}{\mu_{ir}} (1 - e^{-\mu_{ir}}) \right)^{S_i} \right], \quad (B.4.4)$$

where the  $\mu_{ir}$  are defined by equation (B.4.2) and  $(1 - e^{-0})/0$  is interpreted as 1, as just explained.

If the conditions (B.1.1) hold, i.e., each parking area contains exactly one target, then equation (B.4.4) reduces to equation (13) of Reference [9]. Note that Reference [9] derives an error bound for the approximation of its equation (13) to the exact equation. This derivation could probably be extended to the parking areas case, but this has not been done here (and the resulting error bound might be large).

A somewhat simpler approximate attrition equation can be developed by replacing each  $\mu_{ij}$  with

$$\bar{\mu}_i = \sum_{s=1}^n d_{is} T_s, \quad (B.4.5)$$

yielding

$$T_j^K \approx T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij} (1 - e^{-\bar{\mu}_i})}{M_{a(j)} \bar{\mu}_i} \sum_{r \in U_{a(j)}} d_{ir} T_r \right)^{S_i} \right], \quad (B.4.6)$$

where, again,  $(1 - e^{-0})/0$  is interpreted as 1 (although  $\bar{\mu}_i = 0$  implies that type-*i* searchers are completely ineffective, and if  $\bar{\mu}_i = 0$ , the indicated summation in (B.4.6) is also zero and the *i*<sup>th</sup> product term is 1). If the "no-parking-areas" conditions (B.1.1) hold, (B.4.6) reduces to equation (14) of Reference [9]. Reference [9] derives an error bound for the approximation of its equation (14) to the exact case; the derivation could probably be extended to the parking areas case, but this has not been performed here.

### b. An Approximate General Attrition Equation of Binomial Form

To round out the section on attrition processes with parking areas and uniform allocation of fire, we present an approximate equation for the expected number of targets of type  $j$  killed which:

- does not contain exponential terms,
- accepts general detection probabilities  $d_{ir}$ , functions of both searcher type and target type,
- is relatively simple to evaluate, and
- is exact for all the special cases discussed in Sections B.2 and B.3.

To develop this attrition equation, first note that, for each  $i$  and  $r$ ,

$$e^{-d_{ir}T_r} \approx (1 - d_{ir})^{T_r}, \quad (B.4.7)$$

thus

$$e^{-\bar{\mu}_i} \approx \prod_{r=1}^n (1 - d_{ir})^{T_r}, \quad (B.4.8)$$

where  $\bar{\mu}_i$  is defined by equation (B.4.5). One can "unexponentiate" the Poisson approximation by substituting (B.4.8) and (B.4.5) in (B.4.6). This yields the equation

$$T_j^K \approx T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{M_{a(j)}} \left[ \frac{\sum_{r \in U_{a(j)}} d_{ir} T_r}{\sum_{r=1}^n d_{ir} T_r} \right] \left[ 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \right] \right)^{S_i} \right] \quad (B.4.9)$$

If there is an  $i$  such that

$$\bar{\mu}_i \equiv \sum_{r=1}^n d_{ir} T_r = 0$$

(assuming that all parameters are nonnegative and all numbers of targets  $T_r$  are integer), then either  $d_{ir}=0$  or  $T_r=0$  or both, for each  $r$ , i.e., type- $i$  searchers can detect none of the targets present. In this case, the  $i^{\text{th}}$  term of the outer indicated product in (B.4.9) should be regarded as 1, as a target of any type that is present will certainly survive such a searcher.

Error bounds on the approximation (B.4.7) could probably be used in conjunction with the methods of Reference [9] to develop overall error bounds for the approximation (B.4.9) to the (unknown) exact  $T_j^K$ , but this has not been done here, and the resulting bounds might be large. It can be verified, however, that equation (B.4.9) reduces to the exact attrition equation in the special cases discussed above. Specifically:

- If the  $d_{ij}$  satisfy condition (B.2.1) then (B.4.9) reduces to the attrition equation defined by equations (B.2.6) and (B.2.7) of Section B.2.a ( $\bar{d}_i$  or zero case);
- If the  $d_{ij}$  satisfy condition (B.2.1) and all sets  $J_i$  are equal to  $\{1, \dots, n\}$  then (B.4.9) reduces to equation (B.2.8) of Section B.2.b ( $\bar{d}_i$  only case);
- If  $A=1$  (which implies that  $a(j)=1$  and  $U_{a(j)}=\{1, \dots, n\}$  for each  $j$ ) then (B.4.9) reduces to equation (B.3.2) of Section B.3.a (ATRTAB Option 3 case); and
- If the  $d_{ij}$  satisfy condition (B.3.4), then (B.4.9) reduces to equation (B.3.9) of Section B.3.b (ATRTAB Option 2 case).

If the "no-parking-areas" conditions (B.1.1) hold, (B.4.9) reduces to the expression

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij} d_{ij}}{\bar{\mu}_i} \left[ 1 - \prod_{r=1}^n (1 - d_{ir})^{T_r} \right] \right)^{S_i} \right], \quad (B.4.10)$$

where  $\bar{\mu}_i$  is as defined in (B.4.5). This expression provides an approximate attrition equation in the no-parking-areas case with general  $d_{ij}$ ; it is an alternative to equation (17) of Reference [9]. Reference [14] provides additional discussion of this equation.

To repeat, Section B has explored combat processes with "uniform allocation of fire," where each searcher chooses its primary target uniformly from the targets it has detected. Section C examines a combat process with a different rule for target choice.

## C. PARKING AREAS AND STRICT PRIORITY ALLOCATION OF FIRE

### 1. Introduction and Assumptions

Let the parameters  $n$ ,  $m$ ,  $T_j$ ,  $S_i$ ,  $U_a$ ,  $d_{ij}$ , and  $k_{ij}$  be the same as in Section B.1.a. Suppose that there are  $m$  permutation mappings  $\psi_1, \dots, \psi_m$ , (one for each searcher type), each operating on the set of target types  $\{1, \dots, n\}$ . These mappings are to be interpreted as "targets of type  $\psi_i(1)$  are the  $i^{\text{th}}$  priority for searchers of type  $i$ ," where lower  $i$  correspond to higher priority. (E.g., targets of type  $\psi_2(1)$  are the highest priority targets for type-2 searchers.) With the parameters as just described, consider a combat process that proceeds according to Assumptions (1), (2), (3), (4), and (6) of Section B.1.a, but where Assumption (5) is replaced by Assumption

(5') A searcher of type  $i$  that detects one or more targets chooses exactly one of these targets in such a way that the chosen target belongs to the highest priority type of the targets actually detected by that searcher. If more than one target of the highest priority type is detected, one target is chosen randomly and uniformly from among all those of that type detected. The searcher makes an attack on the parking area that contains the chosen target. (A searcher that makes no detections makes no attack.)

Assumption (5') will be called the "strict priority allocation of fire rule." It is "strict" in the sense that a searcher cannot be indifferent, regarding choice of primary target, between targets of different types--as reflected in the fact that the  $\psi_i$  are permutation mappings. It is also "strict" in that a searcher can never fire at a lower priority target if it has detected a higher priority one, even though, in reality, particular lower priority targets may occasionally draw fire away from higher priority ones because the lower priority ones are occupying particularly valuable territory, or for other reasons. (This rule is discussed more fully in Section B of Reference [12].)

To further illustrate the meaning of the  $\psi_i$ , consider the following example, adapted from Reference [8]. Suppose that  $n=5$  and that for searchers of a given type  $i$ , we have  $\psi_i = (3, 1, 5, 4, 2)$ . That is, targets of type 3 are of the highest priority, followed by targets of type 1, and so on, with targets of type 2 of lowest priority. If the numbers of detected targets by a specific searcher of type  $i$  are 0, 5, 0, 8, 4 for targets of types 1, ..., 5, respectively, then the highest priority targets detected are of type 5 and (conditional on these numbers of detections) each of the four detected targets of type 5 will be chosen with probability 1/4 to be the "primary target" of that particular searcher's attack.

Let  $T_j^K$  denote the expected number of type-j targets killed. It is perhaps paradoxical that an exact and algebraically tractable expression for  $T_j^K$  can be derived in the strict priority allocation of fire case with general  $d_{ij}$ , even though it was not possible to derive such an expression in the "simpler" uniform allocation of fire case. An (exact) attrition equation for the no-parking-areas case and strict priority allocation of fire has been developed in Reference [8]; Section C.2 below does the same for the parking areas case. The first attrition option of Subroutine ATRTAB is a straightforward special case of the strict priority allocation of fire process. To complete the presentation of the ATRTAB options, this special case is described in detail in a separate section (C.3).

## 2. Derivation of Attrition Equation

Assumption (4) and the appropriate methods of Reference [11] hold in the "priority" as well as the "uniform" allocation of fire process, thus Lemma 1 still holds, i.e.,

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m (1 - H_{ij})^{S_i} \right],$$

where  $H_{ij}$  is the probability that a specific type-i searcher kills a specific type-j target. Therefore consider specific searcher  $\sigma$  of type  $i$  and specific target  $\tau$  of type  $j$ , which is located on parking area  $\alpha$ , of type  $a(j)$ . Let the events  $K$ ,  $F_\alpha$ ,  $G_a$ , and  $D$  be as defined in Section B.1.b, namely:

$K$ -- $\sigma$  kills  $\tau$  (so  $P(K)=H_{ij}$ ) ,

$F_\alpha$ -- $\sigma$  attacks parking area  $\alpha$ ,

$G_a$ -- $\sigma$  attacks a parking area of type  $a$ , and

$D$ -- $\sigma$  detects at least one target.

In addition, for each target type  $r=1,\dots,n$  define the event

$D_r$ --searcher  $\sigma$  detects at least one target of type  $r$  and no targets the types of which are of higher priority (for searchers of type  $i$ ) than type  $r$ .

It is evident that

$$D = \bigcup_{r=1}^n D_r \quad (C.2.1)$$

and that this union is disjoint. If  $D_r$  occurs, then the "primary target" of  $\sigma$ 's attack will be of type  $r$ . The sets  $U_a$  partition  $\{1,\dots,n\}$ , i.e., no type of target can be located on more than

one type of parking area. Also, type- $r$  targets are located on the (unique) type  $a(r)$  of parking area. Thus

$$D_r \subset G_{a(r)}$$

and

$$D_r \cap G_{a'} = \emptyset \quad \text{for } a' \neq a(r).$$

Recall that the specific target  $\tau$  being considered is of type  $j$ ; by the definition of the function  $a$ , for any target type  $r$ ,  $a(r) = a(j)$  precisely when  $r \in U_{a(j)}$ . Thus

$$\begin{aligned} P(G_{a(j)}) &= P(G_{a(j)} | \bar{D}) P(\bar{D}) + \sum_{r=1}^n P(G_{a(j)} | D_r) P(D_r) \\ &= \sum_{r \in U_{a(j)}} P(D_r). \end{aligned} \quad (\text{C.2.2})$$

Equation (B.1.6) holds here, since different parking areas of the same type are assumed to be identical, i.e.,

$$P(F_\alpha | G_{a(j)}) = 1/M_{a(j)} \quad (\text{C.2.3})$$

(and  $P(F_\alpha | G_a) = 0$  for  $a \neq a(j)$ ), and because of Assumption (6), equation (B.1.3), i.e.,

$$H_{ij} \equiv P(K) = k_{ij} P(F_\alpha) \quad (\text{C.2.4})$$

also holds here.

Note that the priority ranking  $\Psi_i$  is a one-to-one function from  $\{1, \dots, n\}$  onto itself and that its inverse  $\Psi_i^{-1}$  has the interpretation that  $\Psi_i^{-1}(r)$  is the priority, for searchers of type  $i$ , of targets of type  $r$ . Therefore, by Assumptions (2) and (3),

$$P(D_r) = \left[ 1 - (1 - d_{ir})^{T_r} \right] \prod_{l=1}^{\Psi_i^{-1}(r)-1} \left( 1 - d_{il} \Psi_i(l) \right)^{T_{\Psi_i(l)}}. \quad (\text{C.2.5})$$

For those pairs  $(i, r)$  where  $r = \Psi_i(1)$ , so  $\Psi_i^{-1}(r) = 1$ , the indicated product in (C.2.5) runs from 1 to 0 and is thus meaningless. In this case, however, type- $r$  targets are the highest

priority for type- $i$  searchers, thus  $D_r$  occurs precisely when searcher  $\sigma$  (of type  $i$ ) detects at least one type- $r$  target. Thus if  $r = \psi_i(1)$ ,

$$P(D_r) = 1 - (1 - d_{ir})^{T_r}.$$

Expression (C.2.5) is therefore reasonable in all cases if the indicated product is interpreted as unity when  $\psi_i^{-1}(r) = 1$ .

It is straightforward to verify that  $\sum_{r=1}^n P(D_r) = P(D)$ , where  $P(D)$  is as defined in (B.1.4).

Combining expressions (C.2.5), (C.2.2), (C.2.3), (C.2.4), and Lemma 1 yields the exact attrition equation

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{M_{a(j)}} \sum_{r \in U_{a(j)}} \left[ \left[ 1 - (1 - d_{ir})^{T_r} \right] \left[ \prod_{t=1}^{\psi_i^{-1}(r)-1} (1 - d_{i,\psi_i(t)})^{T_{\psi_i(t)}} \right] \right] \right) \right]. \quad (C.2.6)$$

It can be verified that if the "no parking areas" conditions (B.1.1) hold, then equation (C.2.6) reduces to the equation given in the "Proposition" of Reference [8].

The reasoning of Reference [8] is somewhat different from the arguments here; Reference [8] does not use a symmetry argument such as equation (C.2.3) (which is used in the no-parking-areas case in Reference [12]) but conditions on the total number of targets searcher  $\sigma$  detects (cf. Reference [11]).

The strict priority allocation of fire rule may or may not be more realistic than the uniform allocation of fire rule, and some other rule may be more realistic than either of these; see, for example, the discussion in Reference [12].

### 3. A Specific Case--ATRTAB Option 1

The first option for computing attrition in Subroutine ATRTAB is a simple special case of equation (C.2.6). Only two types of targets are considered: open (nonsheltered) aircraft and aircraft shelters. The searchers are enemy aircraft attacking the airbase; the number of types of searchers can be a generic  $m$ . Open aircraft may be located on parking

areas with other open aircraft, but an attack on an aircraft shelter can kill only that shelter. The attack protocol is: if a searcher detects any open aircraft, it picks one at random (uniformly) from those it has detected and fires at its parking area; if a searcher detects no open aircraft but some aircraft shelters, it picks a shelter at random from those it has detected and fires at it.

In the notation of this paper, this situation can be considered a special case of the strict priority allocation of fire process, with the following conditions on the parameters:

$$\left. \begin{array}{l} n=2 \\ A=2 \\ U_a=\{a\} \quad \text{for } a=1,2 \\ \text{(thus } a(j)=j \text{ for } j=1,2) \\ M_2=T_2 \\ \psi_i(j)=j \text{ for } j=1,2 \text{ and } i=1, \dots, m. \end{array} \right\} \quad (C.3.1)$$

That is, open aircraft are considered the type-1 targets and aircraft shelters, the type-2 targets. Substituting into equation (C.2.6) for  $j=1$  and  $j=2$ , respectively, yields the expected number of open aircraft killed,

$$T_1^K = T_1 \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{i1}}{M_1} \left[ 1 - (1 - d_{i1})^{T_1} \right] \right)^{S_i} \right], \quad (C.3.2)$$

and the expected number of aircraft shelters killed,

$$T_2^K = T_2 \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{i2}}{T_2} \left[ 1 - (1 - d_{i2})^{T_2} \right] \left[ (1 - d_{i1})^{T_1} \right] \right)^{S_i} \right]. \quad (C.3.3)$$

#### D. PROCESSES IN WHICH A SEARCHER DETECTS PARKING AREAS

In the combat processes examined in Sections B and C, the object of a searcher's attack is a parking area, but the searcher detects targets individually, and the parking area a searcher attacks is chosen via a random choice of detected target. This section explores an analogous but different set of combat processes, in which a searcher detects parking areas, and chooses one parking area to attack from those it has detected; every target on the

attacked parking area can be killed. These processes are thus closer to an "area fire" model. A series of exact and approximate expressions for the expected number of targets killed can be derived; these expressions parallel the series of attrition equations derived in Sections B and C, but have somewhat different forms. These expressions appear in Sections D.2 and D.3; Section D.1 presents the appropriate preliminaries. For conciseness, proof and derivation details have been omitted; they are similar to the methods of Sections B and C above, and of References [5], [9], and [11].

The parallels between the equations presented here and those derived in Sections B and C are heightened by the fact that if the parameters satisfy the "no-parking-areas" conditions (B.1.1), then each equation in Section B or C and its corresponding equation in Section D.2 or D.3 reduce to the same "no-parking-areas" equation. This, of course, follows from the fact that the underlying combat processes become identical. Section D.4 presents the details.

### 1. Terminology, Assumptions, and Proof Elements

Let the following input parameters be the same as in Section B.1.a:

- $m$  = number of types of searchers,
- $n$  = number of types of targets,
- $A$  = number of types of parking areas,
- $S_i$  = number of searchers of type  $i$  ( $i=1,\dots,m$ ),
- $T_j$  = number of targets of type  $j$  ( $j=1,\dots,n$ ),
- $M_a$  = number of parking areas of type  $a$  ( $a=1,\dots,A$ ), and
- $U_a$  = the subset of  $\{1,\dots,n\}$  that gives the types of targets that are located on type- $a$  parking areas.

It is (still) assumed that the sets  $U_a$  partition the set of target types  $\{1,\dots,n\}$ , and that all parking areas of a given parking area type contain identical complements of targets. (Some of the sets  $U_a$  could be empty, corresponding to parking areas that contain no targets; it might be realistic to use this option in some scenarios.) Thus the following parameters are also the same as in Section B.1.a:

- $a(j)$  = the (unique) type of parking area on which type- $j$  targets are located,
- $M_{a(j)}$  = the number of parking areas that can accommodate type- $j$  targets,  
and
- $t_j$  =  $T_j/M_{a(j)}$  = the number of type- $j$  targets on each type- $a(j)$  parking area.

It is necessary to assume that, for all  $j$ ,  $M_{a(j)} \geq 1$  if  $T_j > 0$ . Strictly speaking, the derivations are valid only if all the  $S_i$ ,  $M_a$ , and  $t_j$  are nonnegative integers, but the resulting formulas are sensible with noninteger values of these parameters if all nonzero  $M_a$  and  $t_j$  are greater than or equal to unity (see Section B.1.a, above).

With the parameters as just described, consider a combat process that proceeds according to the following assumptions.

- (1) At a fixed time, all parking areas (and the targets located on them) become vulnerable to detection and attack.
- (2) Any particular searcher of type  $i$  detects any particular parking area of type  $a$  with probability  $q_{ia}$ .
- (3) Detections of different parking areas by a given searcher are mutually independent events.
- (4) The detection and attack processes of different searchers are mutually independent.
- (5) Of the parking areas it has detected, each searcher chooses one parking area according to a uniform distribution, and makes an attack on that parking area. (A searcher that makes no detections makes no attack.)
- (6) If an attack by a type- $i$  searcher is made on a given parking area, then each type- $j$  target located on that parking area is killed with probability  $k_{ij}$ . The effects of different attacks on the same parking area are independent.

As indicated earlier, these assumptions differ from those of Section B.1.a in that each searcher detects and attacks parking areas, rather than individual targets. It is still desired, however, to compute (for each  $j$ ) the expected number of type- $j$  targets (not parking areas) killed. Let  $T_j^K$  denote this quantity. Since Assumptions (4) and (6) are the same as before, Lemma 1 still holds, i.e.,

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m (1 - H_{ij})^{S_i} \right], \quad (D.1.1)$$

where  $H_{ij}$  is the probability that a specific type- $i$  searcher kills a specific type- $j$  target (which might also be killed by other searchers). Consider a specific searcher  $\sigma$ , of type  $i$ , and a specific target  $\tau$ , of type  $j$ , which is located on parking area  $\alpha$  (which is thus of type  $a(j)$ ). Equation (B.1.3) continues to hold here, i.e.,

$$H_{ij} = k_{ij} P(F_\alpha) \quad (D.1.2)$$

where  $F_\alpha$  is the event that searcher  $\sigma$  attacks parking area  $\alpha$ . Furthermore, by Assumptions (2), (3), and (5),

$$P(F_\alpha) = \sum_{x=0}^{\infty} \frac{q_{i,\alpha(j)}}{x+1} P(X=x), \quad (D.1.3)$$

where the random variable  $X$  represents the number of parking areas, other than parking area  $\alpha$ , that searcher  $\sigma$  detects. (Equation (D.1.3) is comparable to equation (B.4.1).) The attrition equations in the next section are all derived by finding exact or approximate expressions for the probability distribution of  $X$  and applying equations (D.1.3), (D.1.2), and (D.1.1).

Assumption (5) could conceivably imply the attack of an empty parking area--if for some  $a$  and  $i$ ,  $M_a > 0$  and  $q_{ia} > 0$ , but  $T_j = 0$  for all  $j \in U_a$ . This situation can be avoided, if desired, by preprocessing the data so that  $M_a$  is set equal to zero for all parking area types  $a$  where  $T_j = 0$  for all  $j \in U_a$  (so that, in effect, no parking areas are empty). (This preprocessing can also be performed, if desired, when the "strict priority allocation of fire" process of Section D.3, below, is used.)

## 2. Equations for the Uniform Allocation of Fire Cases

All the equations in this section follow from Assumptions (1) through (6) above, with additional assumptions as indicated.

### a. Nonzero Detection Probabilities a Function of Searcher Type Only

Suppose that the detection probabilities  $q_{ia}$  satisfy the following property:

$$\left. \begin{array}{l} \text{For each } i, \text{ there exists a value } \bar{q}_i \in [0,1] \\ \text{and a subset } Q_i \text{ of } \{1, \dots, A\} \text{ such that} \\ \quad \text{--for all } a \in Q_i, q_{ia} = \bar{q}_i \text{ and} \\ \quad \text{--for all } a \notin Q_i, q_{ia} = 0. \end{array} \right\} \quad (D.2.1)$$

This is analogous to condition (B.2.1); some or all of the sets  $Q_i$  could be empty. In this case, it can be proved that the expected number of type- $j$  targets killed is given by

$$T_j^K = T_j \left[ 1 - \prod_{i \in R_j} \left( 1 - \frac{k_{ij}}{\bar{W}_i} \left[ 1 - (1 - \bar{q}_i)^{\bar{W}_i} \right] \right)^{S_i} \right] \quad (D.2.2)$$

where

$$\bar{W}_i = \sum_{a \in Q_i} M_a \quad i = 1, \dots, m, \quad (D.2.3)$$

and

$$R_j = \{i \mid a(j) \in Q_i \text{ and } \bar{W}_i > 0\}. \quad (D.2.4)$$

This equation is (in some sense) analogous to equations (B.2.6) and (B.2.7). An analogy to ATRTAB Option 2 can be developed as a special case of (D.2.2). In ATRTAB Option 2, searchers of a given type can only (detect and) attack one type of parking area. In the context of the process where searchers detect whole parking areas, this feature corresponds to each set  $Q_i$  having exactly one element, i.e., for each  $i$ ,  $q_{ia}$  is nonzero for (at most) one value of  $a$ .

**b. All Detection Probabilities a Function of Searcher Type Only**

If the detection probabilities  $q_{ia}$  satisfy condition (D.2.1) and in addition, each set  $Q_i$  is equal to the whole set  $\{1, \dots, A\}$  of parking area types, then detection probabilities are a function of searcher type only. In this case, the expected number of targets killed is given by

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{M} [1 - (1 - \bar{q}_i)^M] \right)^{S_i} \right], \quad (D.2.5)$$

where

$$M = \sum_{a=1}^A M_a \quad (D.2.6)$$

denotes the total number of parking areas. Equation (D.2.5) is analogous to equation (B.2.8). The special case of Equation (D.2.5) when  $A=1$  can be considered as being analogous to the ATRTAB Option 3 equation (B.3.2).

**c. Approximate Attrition Equations With General Detection Probabilities**

In analogy with Section B.4, this section presents three expressions which represent reasonable approximations, if not exact values, for the expected number of targets

killed,  $T_j^K$ . These expressions accept general detection probabilities  $q_{ia}$ . Consider a specific searcher  $\sigma$ , of type  $i$ , and a specific target  $\tau$ , of type  $j$ , which is located on parking area  $\alpha$ , of type  $a(j)$ . The first expression utilizes the fact that  $X$ , the number of parking areas other than area  $\alpha$  that searcher  $\sigma$  detects, is approximately Poisson distributed with mean

$$v_{ij} = q_{i,a(j)}(M_{a(j)} - 1) + \sum_{b \neq a(j)} q_{ib} M_b. \quad (D.2.7)$$

Applying equations (D.1.3), (D.1.2), and (D.1.1) yields

$$T_j^K \approx T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij} q_{i,a(j)}}{v_{ij}} [1 - e^{-v_{ij}}] \right)^{S_i} \right] \quad (D.2.8)$$

This approximation stands in analogy to expression (B.4.4), and as there, if  $v_{ij} = 0$ ,  $(1 - e^{-v_{ij}})/v_{ij}$  should be interpreted as unity.

Replacing  $v_{ij}$  in equation (D.2.8) with

$$\bar{v}_i = \sum_{b=1}^A q_{ib} M_b \quad (D.2.9)$$

yields the approximation

$$T_j^K \approx T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij} q_{i,a(j)}}{\bar{v}_i} [1 - e^{-\bar{v}_i}] \right)^{S_i} \right], \quad (D.2.10)$$

which is analogous to expression (B.4.6). As before, if  $\bar{v}_i = 0$ ,  $(1 - e^{-\bar{v}_i})/\bar{v}_i$  should be interpreted as unity. (If the condition holds that  $M_{a(j)} > 0$  whenever  $T_j > 0$ , then  $T_j > 0$  and  $\bar{v}_i = 0$  implies that  $q_{i,a(j)} = 0$ , and the  $i$ th term of the indicated product is unity.)

Using the approximation

$$e^{-\bar{v}_i} = \prod_{b=1}^A (1-q_{ib})^{M_b} \quad (D.2.11)$$

and substituting in (D.2.8) yields

$$T_j^K \approx T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij} q_{i,a(j)}}{\bar{v}_i} \left[ 1 - \prod_{b=1}^A (1-q_{ib})^{M_b} \right]^{S_i} \right) \right], \quad (D.2.12)$$

which is analogous to expression (B.4.9). (If  $\bar{v}_i=0$ , regard the  $i^{\text{th}}$  term of the outer indicated product as unity.) It can be verified that (D.2.12) reduces to (D.2.2) or (D.2.5) if the appropriate conditions ((D.2.1) or (D.2.1) plus the added condition that all  $Q_i=\{1,\dots,A\}$ , respectively) hold.

### 3. Equation for the Strict Priority Allocation of Fire Case

In analogy to Section C, one can consider a combat process where a searcher detects parking areas and chooses to attack a parking area of the highest priority type it has detected. Let there be  $m$  permutation mappings  $\phi_i$  (one for each searcher type), operating on the set  $\{1,\dots,A\}$  of parking area types, to be interpreted as "parking areas of type  $\phi_i(1)$  are the  $1^{\text{th}}$  priority for searchers of type  $i$ ." As in Section C, lower  $i$  correspond to higher priority. Let the parameters  $m$ ,  $n$ ,  $A$ ,  $S_i$ ,  $T_j$ ,  $M_a$ ,  $U_a$ ,  $a(j)$ ,  $q_{ia}$ , and  $k_{ij}$  be the same as in Section D.1; consider a combat process that proceeds according to Assumptions (1), (2), (3), (4), and (6) of Section D.1, but where Assumption (5) is replaced by Assumption

(5'): A searcher of type  $i$  that detects one or more parking areas chooses exactly one of these parking areas in such a way that the chosen parking area belongs to the highest priority type of the parking areas actually detected by that searcher. If more than one parking area of the highest priority type is detected, one parking area is chosen randomly and uniformly from among all those of that type detected. The searcher makes an attack on the chosen parking area. (A searcher that makes no detections makes no attack.)

It can be shown that an exact expression for the expected number of type- $j$  targets killed in this combat process is

$$T_j^K = T_j \left[ 1 - \prod_{i=1}^m \left( 1 - \frac{k_{ij}}{M_{a(j)}} \left[ 1 - (1 - q_{i,a(j)}) M_{a(j)} \right] \prod_{v=1}^{\phi_i^{-1}(a(j))-1} (1 - q_{i,\phi_i(v)}) M_{\phi_i(v)} \right) \right]. \quad (D.3.1)$$

As in Section C, for those pairs  $(i, j)$  where  $a(j) = \phi_i(1)$ , the inner indicated product is to be interpreted as unity.

If equation (D.3.1) is implemented in a combat model, then the quantities used by the equation can be computed in the model, based on other inputs--they need not be direct inputs to the model. In particular, the priority orderings  $\phi_i$  can be computed by ranking the parking area types according to some desired measure of value, which can perhaps be a function of the numbers and types of targets on these areas. The effect of this might be that a searcher attacks (from among those parking areas it detects) the parking area with the most targets on it, or the parking area with a maximal weighted number of targets (with input weights that give relative values for targets of different types), or the parking area that contains the highest priority targets (as determined by some function  $\psi$  for target priority, cf. Section C, above).

#### 4. Reduction to the "No-Parking-Areas" Case

Suppose the parameters satisfy the following conditions (which are identical to the conditions (B.1.1) of Section B):

$$\left. \begin{array}{l} A = n \\ a(j) = j \quad \text{for } j = 1, \dots, n, \\ U_{a(j)} = \{j\} \quad \text{for } j = 1, \dots, n, \text{ and} \\ M_{a(j)} = T_j \quad \text{for } j = 1, \dots, n. \end{array} \right\} \quad (D.4.1)$$

Then each parking area contains exactly one target, and the combat processes of Sections D.1 and D.3 become identical to the "basic" combat processes, without parking areas, of References [5] (Chapter III) and [8], respectively. So do the processes of Sections B and C, as was indicated in those sections. One would thus expect corresponding attrition equations to reduce to the same "no-parking-areas" equation, and this indeed does occur.

For example, the expression for  $T_j^K$  given by (B.2.6) and (B.2.7) appears quite different from expression (D.2.2). Yet if the conditions (D.4.1) hold (and the appropriate

notation changes are made) both expressions for  $T_j^K$  reduce to the same equation. This "reduced" equation appears as equation (12) of Reference [11] and equation (21) of Reference [9]; it can compute attrition in the case where targets are not considered as being located on parking areas and where detection probabilities, if not zero, are a function only of searcher type.

The situation is similar for the other equations. Specifically, if the conditions (D.4.1) (i.e., (B.1.1)) hold, then:

1. (B.2.6)/(B.2.7) and (D.2.2) both reduce to equation (21) of Reference [9] ( $\bar{d}_i$  or 0 case, as stated above);
- 1a. If the condition (B.3.4) also holds, then (B.3.9) and the special case of (D.2.2) where each set  $Q_i$  contains one element both reduce to the special case of equation (21) of Reference [9] where each set  $J_i$  has exactly one element, i.e., each type of searcher can detect and attack only one type of target (ATRTAB Option 2 case);
2. (B.2.8) and (D.2.5) both reduce to equation (3.14) of Reference [6], which also appears in several other references, as indicated in Section B.2.b, above ( $\bar{d}_i$  only case);
- 2a. (B.3.2) with  $n=1$  and the special case of (D.2.5) where  $A=1$ , so that  $\bar{q}_i=q_{i1}$  for all  $i$ , both reduce to the special case of equation (3.14) of Reference [6] where there is only one type of target (ATRTAB Option 3 case);
3. (B.4.4) and (D.2.8) both reduce to equation (13) of Reference [9] (first Poisson approximation case);
4. (B.4.6) and (D.2.10) both reduce to equation (14) of Reference [9] (second Poisson approximation case);
5. (B.4.9) and (D.2.12) both reduce to equation (B.4.10) of the current paper, which is also equation (B.3.1) of Reference [14] (general  $d_{ij}$  binomial approximation); and
6. (C.2.6) and (D.3.1) both reduce to the equation given in the "Proposition" of Reference [8] (strict priority allocation of fire case).

## E. CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

This paper has derived a number of attrition equations that treat the situation where an attack on one target can kill other targets located in the same area, under varying assumptions about the detection and attack processes. The work has been related to 1) a procedure used in several combat simulations to model attrition caused by airbase attack

and 2) several attrition equations derived for the no-parking-areas case. This section presents some possible extensions of the work on attrition processes with parking areas.

A series of different combat processes can be generated by varying the specification of targets killed on a parking area, given attack; one could then try to find formulas for the expected numbers of targets killed for these processes. One possibility is:

1. N targets (if there are that many) are selected at random from the parking area that a searcher attacks, and only those targets are vulnerable to that attack, where N is an input (possibly unity).

The parking area that a searcher attacks could be chosen according to (the first five assumptions of) any one of the processes of Section B, C, or D (or some other process).

Some other variations, which involve the idea of "primary target" (the detected one that was "chosen"), and are thus not (directly) applicable to the combat process of Section D, are as follows.

2. A searcher attacks its primary target with some input probability; with one minus that probability, the searcher attacks some other target (where the choice of target is made in some specified manner) on the same parking area as the primary target. Only the attacked target can be killed; the probability of kill given attack can depend on whether the attacked target is the primary target or not.
3. Every target on the attacked parking area is vulnerable, but the primary target is killed with higher (input) probability.
4. The primary target and exactly  $N-1$  other targets (chosen in some specified manner) on the attacked parking area are vulnerable; the probability of kill given attack may be different for the primary target (N is an input).

Item 4 is in some sense a combination of Items 1 and 3.

Some other possible topics for future work are as follows.

5. Develop an attrition equation for the case where target choice is based on a "nonstrict priority allocation of fire" rule, in which an attacker might be indifferent between targets (or parking areas) of some different types.
6. Develop an attrition equation for the case of "weighted allocation of fire," as discussed in Reference [12], where target choice is based on a set of input weights for targets (or parking areas) of different types. (An exact, succinct expression for the expected number of targets killed might not be possible, but there could be approximate or heuristically reasonable expressions.) Related to this is

7. Adapt the "relatively general attrition equation" of Reference [1] to also treat the case of parking areas.
8. The derivations in this paper have frequently utilized the assumption that the sets  $U_a$  partition the set of target types  $\{1, \dots, n\}$ , so that each type of target is located on exactly one type of parking area. Develop attrition equations in the case where targets of a given type might be located on several types of parking areas. (In the uniform allocation of fire case, targets of the same type located on different types of parking areas can simply be considered as targets of different types--with the same detection and kill probabilities--but this doesn't work under priority allocation of fire rules.)
9. Develop error bounds for the approximations (B.4.4), (B.4.6), (B.4.9) (D.2.8), (D.2.10), and (D.2.12) to the expected number of targets killed.

## REFERENCES

- [1] Anderson, L.B., An Initial Postulation of a Relatively General Attrition Process, Working Paper WP-20 of IDA Project 2371, Institute for Defense Analyses, Alexandria, VA, October 1983, revised May 1984.
- [2] Anderson, L.B., J. Bracken, and E.L. Schwartz, Revised OPTSA Model, Volume 1: Methodology, Volume 2: Computer Program Documentation, and Volume 3: The OPTSA Print-Run Program, IDA Paper P-1111, Institute for Defense Analyses, Arlington, VA, October 1975.
- [3] Anderson, L.B., P.A. Frazier, M.J. Hutzler, and F.J. Smoot, Documentation of the IDA Tactical Air Model (IDATAM) Computer Program, IDA Paper P-1409, Institute for Defense Analyses, Arlington, VA, February 1979.
- [4] Anderson, L.B., and E.L. Schwartz, NAVMOD: A Naval Model, Volume 1: Main Report, and Volume 2: Appendices, IDA Report R-278, Institute for Defense Analyses, Alexandria, VA, October 1985.
- [5] Karr, A.F., On a Class of Binomial Attrition Processes, IDA Paper P-1031, Institute for Defense Analyses, Arlington, VA, June 1974.
- [6] Karr, A.F., Lanchester Attrition Processes and Theater-Level Combat Models, IDA Paper P-1528, Institute for Defense Analyses, Arlington, VA, September 1981. Also appears in Shubik, M. (editor), Mathematics of Conflict, North-Holland Systems and Control Series, Volume 6, North-Holland, Amsterdam, The Netherlands, 1983.
- [7] Karr, A.F., Binomial Versions of Lanchester Attrition Processes: Theory and Application to Models of Theater Level Combat, Working Paper WP-23 of IDA Project 2371, Institute for Defense Analyses, Alexandria, VA, December 1981, reissued November 1984.
- [8] Karr, A.F., A Heterogeneous Linear Law Binomial Attrition Process with Priority Allocation of Fire, Working Paper WP-17 of IDA Project 2371, Institute for Defense Analyses, Alexandria, VA, March 1982.
- [9] Karr, A.F., A Heterogeneous Linear Law Binomial Attrition Equation, Working Paper WP-19 of IDA Project 2371, Institute for Defense Analyses, Alexandria, VA, September 1983.
- [10] Karr, A.F., and E.L. Schwartz, Modeling of Attrition in a Barrier Penetration Process with Evenly Spaced Barrier Elements, IDA Paper P-1727, Institute for Defense Analyses, Alexandria, VA, in draft.

- [11] Schwartz, E.L., A Simple Derivation of a Heterogeneous Linear Binomial Attrition Equation, Working Paper WP-24 of IDA Project 3609, Institute for Defense Analyses, Alexandria, VA, August 1982, revised November 1984.
- [12] Schwartz, E.L., A Short Proof of the Basic Binomial Heterogeneous Linear Attrition Equation, with Indications for Extensions, Working Paper WP-21 of IDA Project 2371, Institute for Defense Analyses, Alexandria, VA, November 1983, revised June 1984.
- [13] Schwartz, E.L., An Attrition Process with Parking Areas, Working Paper WP-1 of IDA Project 3661, Institute for Defense Analyses, Alexandria, VA, February 1984.
- [14] Schwartz, E.L., An Exploration of Some Properties of Several Approximate Binomial Heterogeneous Attrition Equations, Working Paper WP-13 of IDA Project 3666, Institute for Defense Analyses, Alexandria, VA, May 1985.

**DISTRIBUTION**  
**IDA PAPER P-2144**  
**ATTRITION PROCESSES WITH PARKING AREAS**

60 Copies

Dr. Jerome Bracken 5 Magnolia Parkway Chevy Chase, MD 20815	1
Dr. Joshua Epstein The Brookings Institution 1775 Massachusetts Ave., N.W. Washington, D.C. 20036	1
Dr. Ray Jakobovits Metron, Inc. 1485 Chain Bridge Road McLean, VA 22101	1
Professor Alan F. Karr Department of Mathematical Sciences The Johns Hopkins University Baltimore, MD 21218	1
Dr. Royce Kneece Office of the Secretary of Defense Room 1E466, The Pentagon	1
Professor Richard M. Soland Department of Operations Research School of Engineering and Applied Science George Washington University Washington, D.C. 20052	1
Office of the Secretary of Defense OUSDRE (DoD-IDA Management Office) 1801 N. Beauregard Street Alexandria, VA 22311	1
Defense Technical Information Center Cameron Station Alexandria, VA 22314	1
National Technical Information Service 5285 Port Royal Road Springfield, VA 22161	1

Distribution List (cont'd)

Page 2

Department of the Air Force Assistant Chief of Staff, Studies and Analyses Rm. 1E388, The Pentagon	1
ANSER Attention: Library 1215 Jefferson Davis Highway Arlington, VA 22202	1
Department of the Army Deputy Under Secretary (Operations Research) Room 2E660, The Pentagon	1
Hudson Institute, Inc. Center for Naval Analyses Attention: Library P.O. Box 16268 Alexandria, VA 22302-0268	1
U.S. Army Concepts Analysis Agency Attention: Library 8120 Woodmont Avenue Bethesda, MD 20814-2797	1
Department of Defense National Defense University Attention: Library Fort McNair Washington, D.C. 20319-6000	1
Department of the Navy Naval Postgraduate School Attention: Library Monterey, CA 93943-5100	1
Office of the Joint Chiefs of Staff Force Structure, Resource, and Assessment Directorate (J-8) Capabilities Assessment Division Room 1D940, The Pentagon	1
The RAND Corporation Attention: Library 2100 M Street, N.W., Washington, D.C. 20037	1

Distribution List (cont'd)  
Page 3

The RAND Corporation Attention: Library P.O. Box 2138 Santa Monica, CA 90406-2138	1
Department of the Air Force Attention: Library Tactical Air Command (TAC) Langley AFB, VA 23665-5001	1
Vector Research, Inc. Attention: Library P.O. Box 1506 Ann Arbor, Michigan 48106	1
Institute for Defense Analyses 1801 N. Beauregard Street Alexandria, VA 22311	
Attention: General W.Y. Smith	1
Phillip Major	1
William Schultis	1
Eleanor Schwartz	8
Lowell Bruce Anderson	1
Robert Anthony	1
Robert Atwell	1
Peter Brooks	1
William Cralley	1
Ronald Enlow	1
Arthur Fries	1
Jeffrey Grotte	1
Edward Kerlin	1
Graham McBryde	1
Frederic Miercort	1
Merle Roberson	1
Mitchell Robinson	1
Alan Rolfe	1
Leo Schmidt	1
Jeffrey Tate	1
Nancy Toma	1
John Transue	1
Control and Distribution	10